Closed-form solutions for a noncentral parabolic potential

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(Received 18 December 1998)

We have found closed-form solutions of the Schrödinger equation for a particle in a noncentral potential given by a two-body Coulomb potential plus a parabolic barrier. These kinds of potentials arise in the context of the three-body Coulomb continuum problem. Here we study the continuum and discrete spectrum eigenfunctions as well as their asymptotic behavior and the associated transition amplitudes.
[S1050-2947(99)06106-5]

PACS number(s): 34.50.Fa, 03.65.Nk

The exact solution for the quantum three-body Coulomb problem (3BCP) remains unknown. In the field of atomic collision physics, many strategies have been developed to deal with systems involving 3BCP. In some particular cases, such as the H_2^+ ion molecule, approximate solutions can be obtained using an adiabatic approach [1]. Basis expansions of the solutions in different sets of coordinates have been found to be suitable for low energy collisions [2,3], while solutions of the Schrödinger equation through separation of variables and *ab initio* methods have been widely used in atomic collisions in the intermediate to high energy regime (see, for example, [4–7]). Furthermore, transformations of the Schrödinger equation into an integral equation have also been proposed [8–10]. On the other hand, there is a great number of two-body problems, which have been solved in closed form. Among them, the two-body Coulomb problem (2BCP) stands out, as it is the basic model for the atom. Then, it is usual to write down approximate solutions for the 3BCP in terms of 2BCP eigenfunctions. One of the most thoroughly used approximations in ion-atom collisions relies on separable functions where the different factors correspond to 2BCP solutions [4,5]. Each pair of particles is considered to interact separately with charges unscreened by the presence of the third particle. Several improvements to this model have been employed to include the dynamics of the three-particle system in the wave functions. Recently, the authors and co-workers have proposed an approximate solution for the 3BCP which can be written as the superposition of two-body problem eigenfunctions [11,12]:

$$\Psi^{\Phi_2}=C\varphi_0^{tp}\sum a_m\varphi_m^t\varphi_m^p. \tag{1}$$

φ_0^{tp} is a 2BCP solution, while φ_m^t and φ_m^p are solutions of a Schrödinger equation for the motion of a particle in a noncentral potential.

In general, noncentral potentials are extremely difficult to solve, because of the system's low level of symmetry. From a physical point of view, fewer symmetries always imply fewer constants of motion and, from a mathematical point of view, fewer coordinate systems in which the problem is separable.

This work deals with the problem of finding φ_m^i ($i=t,p$) in closed form, and analyzing the physics of this particular noncentral potential problem. As this system is closely related to the 3BCP, it is expected that further insight into the latter could be gained from the study of the former. As a by-product, we intend to contribute to the general understanding of the physical and mathematical properties of the solutions for a noncentral potential.

The paper is organized as follows: In the next section, we derive solutions for the noncentral potential both for the continuum as well as for the discrete spectra. From the study of the asymptotic behavior of the solutions found, we obtain and analyze the corresponding scattering amplitudes and differential cross sections. Atomic units (a.u.) are used throughout.

I. THE SCHRÖDINGER EQUATION

Let us consider the (time independent) Schrödinger equation for the motion of a spinless particle of mass μ in a potential $V^{\pm}$:

$$H\psi=E\psi,\quad \left[-\frac{1}{2\mu}\nabla_r^2+V^{\pm}\right]\psi(\mathbf{r})=E\psi(\mathbf{r}), \tag{2}$$

where

$$V^{\pm}=-\frac{|Z|}{r}+\frac{\lambda^2}{r^2(1\pm\hat{\mathbf{k}}'\cdot\hat{\mathbf{r}})},\quad V^{\pm}=V^c+V^p. \tag{3}$$

$V^c=|Z|/r$ is a Coulomb potential and λ is a positive integer. V^p can be seen as a parabolic barrier, $\hat{\mathbf{k}}'$ being a parameter. This potential diverges for $r=0$ and for directions given by $1\pm\hat{\mathbf{k}}'\cdot\hat{\mathbf{r}}=0$. According to its definition, $V^{\pm}$ represents a long-range noncentral potential, see Fig. 1.

We will first derive the solutions for the continuum spectrum. Writing

$$\psi(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}\varphi(\mathbf{r}) \tag{4}$$

and replacing in Eq. (2), we obtain for the distortion $\varphi(r)$

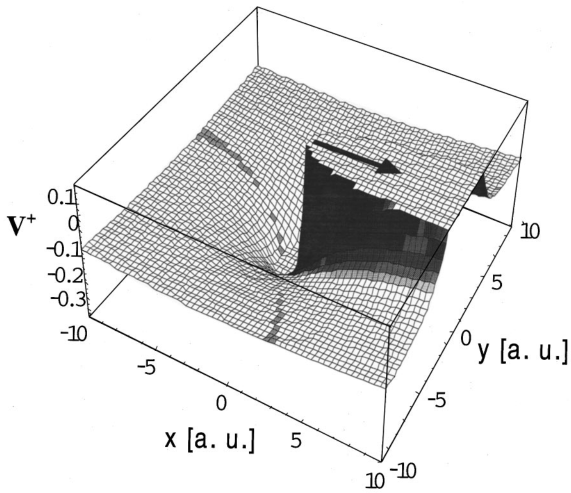

FIG. 1. Noncentral potential V^+ for $\lambda = Z = 1$ a.u., in a plane that includes $\hat{\mathbf{k}}$. The potential is symmetrical around $\hat{\mathbf{k}}$, whose direction is denoted by the arrows.

$$\left[\frac{1}{2\mu}\nabla_r^2 + \frac{i}{\mu}\mathbf{k}\cdot\boldsymbol{\nabla}_r + \frac{1}{r}\left(|Z| - \frac{\lambda^2}{r(1 \pm \hat{\mathbf{k}}'\cdot\hat{\mathbf{r}})}\right)\right]\varphi(\mathbf{r}) = 0. \quad (5)$$

This equation is extremely difficult to solve in general. However, choosing $\hat{\mathbf{k}}' = \hat{\mathbf{k}}$, i.e., the asymptotic particle momentum direction, the parabolic potential can be expressed in terms of the usual parabolic coordinates (ξ, η, ϕ) [13]. In order to perform a detailed analysis, we consider the potential V^+, but solutions can be obtained for V^- in a similar way. Writing Eq. (5) in parabolic coordinates, it reads

$$\frac{1}{\mu r}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + \frac{\mu(\xi+\eta)}{4\xi\eta}\frac{\partial^2}{\partial\phi^2} + ik\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) + \mu\left(|Z| - \frac{\lambda^2}{\xi}\right)\right]\varphi(\mathbf{r}) = 0. \quad (6)$$

This equation can be easily separated into three equations, one for each parabolic coordinate. We will restrict ourselves to the study of ϕ independent solutions for the continuum spectrum. We introduce this separation as

$$\varphi(\mathbf{r}) = \chi_1(\xi)\chi_2(\eta) \quad (7)$$

and Eq. (6) reads

$$\left[\xi\frac{\partial^2}{\partial\xi^2} + (1 + ik\xi)\frac{\partial}{\partial\xi} + \beta_1 - \frac{\lambda^2}{\xi}\right]\chi_1(\xi) = 0, \quad (8)$$

$$\left[\eta\frac{\partial^2}{\partial\eta^2} + (1 - ik\eta)\frac{\partial}{\partial\eta} + \beta_2\right]\chi_2(\eta) = 0, \quad (9)$$

where β_1 and β_2 are separation constants, such that $\beta_1 + \beta_2 = \mu|Z|$.

Solutions to these equations can be written in closed form in terms of confluent hypergeometric functions ${}_1F_1[a,b,z]$ [15], as follows:

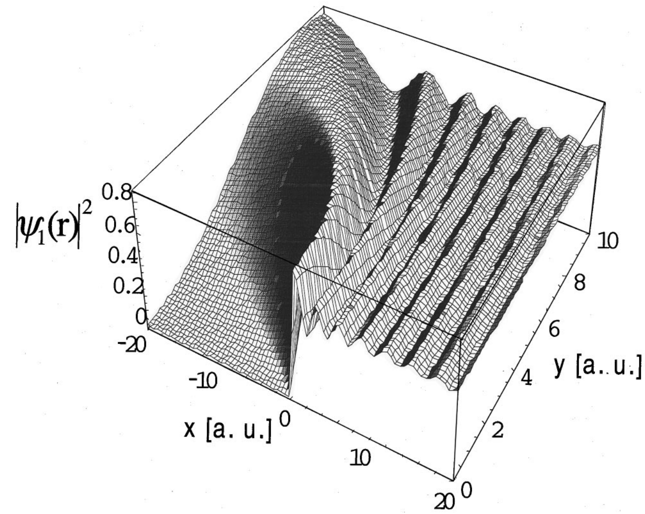

FIG. 2. Particle distribution $|\psi_1(\mathbf{r})|^2$ as a function of the Cartesian coordinates $\mathbf{r} = (x,y,z)$ and for $k = 1$ a.u. and $\lambda = 1$ a.u. Note the probability density removal from the $\hat{\mathbf{k}}$ direction from $r = 0$ to $-\infty$.

$$\chi_1 = (-ik\xi)^{\lambda}\,{}_1F_1\left[-i\frac{\beta_1}{k} + \lambda, 1 + 2\lambda, -ik\xi\right], \quad (10)$$

$$\chi_2 = {}_1F_1\left[i\frac{\beta_2}{k}, 1, ik\eta\right], \quad (11)$$

so that the normalized continuum eigenfunctions reads

$$\psi(\mathbf{r}) = Ne^{i\mathbf{k}\cdot\mathbf{r}}\,{}_1F_1\left[i\frac{\beta_1}{k}, 1, ik\eta\right](-ik\xi)^{\lambda}\,{}_1F_1 \times\left[-i\frac{\beta_2}{k} + \lambda, 1 + 2\lambda, -ik\xi\right], \quad (12)$$

with N being the normalization constant to be determined. We note that, for $\lambda = 0$ Eq. (12) yields the general solution of the 2BCP for $l_z = 0$:

$$\psi = N^c e^{i\mathbf{k}\cdot\mathbf{r}}\,{}_1F_1\left[i\frac{\beta_1}{k}, 1, ik\eta\right]{}_1F_1\left[-i\frac{\beta_2}{k}, 1, -ik\xi\right]. \quad (13)$$

The eigenfunction whose asymptotic behavior represents a plane wave plus an ingoing spherical wave corresponds to the election $\beta_1 = 0$:

$$\psi_1(\mathbf{r}) = Ne^{i\mathbf{k}\cdot\mathbf{r}}[-i(kr + \mathbf{k}\cdot\mathbf{r})]^{\lambda}\,{}_1F_1 \times\left[-i\frac{Z\mu}{k} + \lambda, 1 + 2\lambda, -i(kr + \mathbf{k}\cdot\mathbf{r})\right]. \quad (14)$$

In Fig. 2, we can see the particle distribution obtained by taking the square modulus of $\psi_1(\mathbf{r})$ as a function of the Cartesian coordinates $\mathbf{r} = (x,y,z)$ and for $k = 1$ a.u. and $\lambda = 1$. The effect of the repulsive barrier is to remove the probability density from the $\mathbf{k}$ direction from $r = 0$ to $-\infty$, producing an effect similar to that typically observed in radial wave functions for increasing angular momentum due to the so-called centrifugal barrier. The particle removal in-

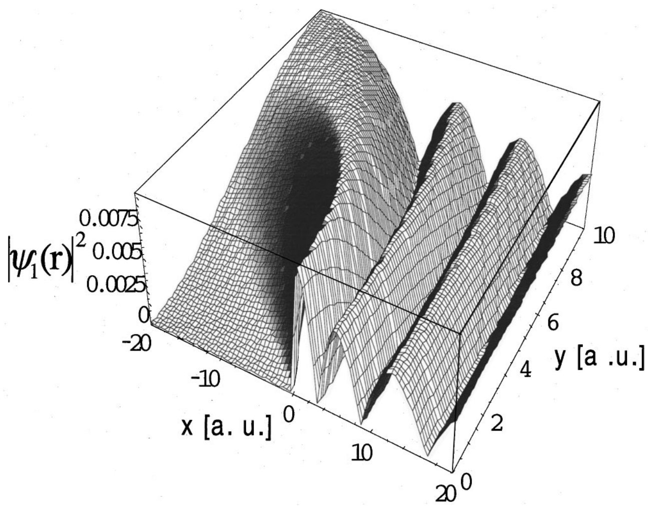

FIG. 3. Same as Fig. 2 for $k=0.01$ a.u.

creases with λ. Similar results are observed for $k=0.01$ a.u. (Fig. 3). For $\lambda=0$, $\psi_1(\mathbf{r})$ reduces to the corresponding 2BCP solution,

$$\psi(\mathbf{r})=N^c e^{i\mathbf{k}\cdot\mathbf{r}}{}_1F_1\left[-i\frac{Z\mu}{k},1,-i(kr+\mathbf{k}\cdot\mathbf{r})\right]. \tag{15}$$

Choosing $\beta_2=0$, we have

$$\psi_2(\mathbf{r})=Ce^{i\mathbf{k}\cdot\mathbf{r}}{}_1F_1\left[i\frac{Z\mu}{k},1,ik\eta\right]\times(-ik\xi)^{\lambda}{}_1F_1[\lambda,1+2\lambda,-ik\xi]. \tag{16}$$

One of the solutions found for Eq. (2), associated with V^+, for which no Coulomb potential is present, is given by

$$\psi_3(\mathbf{r})=N^P e^{i\mathbf{k}\cdot\mathbf{r}}(-ik\xi)^{\lambda}{}_1F_1[\lambda,1+2\lambda,-ik\xi]. \tag{17}$$

By performing an analytical extension to the complex plane of $k=-iZ\mu/n$ in Eq. (12), bound state solutions can also be obtained [14]:

$$\psi_{n_1,n_2,m}(\mathbf{r})=Ne^{-(Z\mu/n)\mathbf{r}}\left(\frac{Z\mu}{n}\xi\right)^{\sqrt{4\lambda^2+m^2}/2}\left(\frac{Z\mu}{n}\eta\right)^{|m|/2}\times{}_1F_1\left[-n_1,1+\frac{\sqrt{4\lambda^2+m^2}}{2},\frac{Z\mu}{n}\xi\right]\times{}_1F_1\left[-n_2,1+|m|,\frac{Z\mu}{n}\eta\right], \tag{18}$$

where n_1, n_2, and m are non-negative integers and N is the normalization constant. Thus each stationary state of the Schrödinger equation (2) is determined by three integers: the parabolic quantum numbers n_1 and n_2, and the magnetic quantum number m. For n, the principal quantum number, we have

$$n=n_1+n_2+1+\frac{|m|}{2}+\frac{\sqrt{4\lambda^2+m^2}}{2}, \tag{19}$$

and the energy results

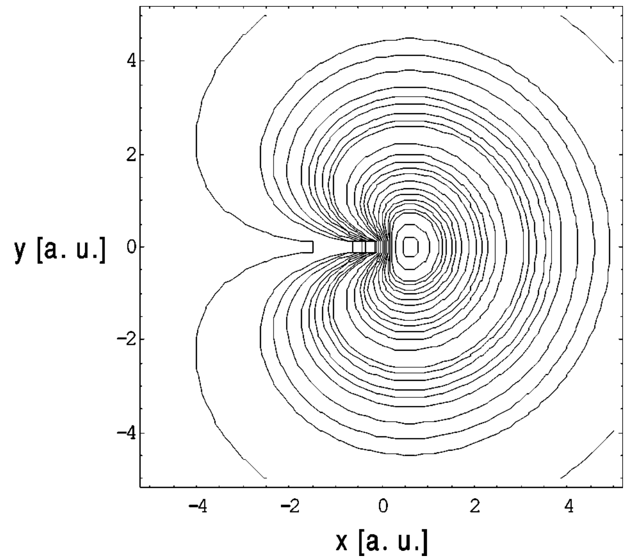

FIG. 4. Particle distribution $|\psi_{n_1=0,\ n_2=0,\ m=0}(\mathbf{r})|^2$ for the ground state as a function of $\mathbf{r}$, in the plane defined by $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$, and for $\lambda=1$ a.u. Note that the repulsive parabolic barrier removes the probability from the direction $\hat{\mathbf{k}}$. Furthermore, the ground state spherical symmetry is lost, and the maximum lies out of the origin.

$$E=-\frac{Z^2\mu}{2n^2}. \tag{20}$$

We should note that n is not in general an integer number, as the square root in Eq. (19) needs not be an integer. For a fixed value of λ and integer values of n_1, n_2, and m we fix n, but we can get the same value of n choosing different n_1, n_2, and m, i.e., n is a degenerate number. For given n, the number $|m|$ can take different values from 0 to $(1+n^2-2n-\lambda^2)/(n-1)$. As we can see in Eq. (19), the lower value of n is $n=1+|\lambda|$. For fixed n and $|\lambda|$ the number n_1 takes $n-|m|/2-\sqrt{4\lambda^2+m^2}/2$ values from 0 to $n-|m|/2-\sqrt{4\lambda^2+m^2}/2-1$.

It is interesting to note that due to the fact that the potential is noncentral, the square of the angular momentum $\mathbf{L}^2$ does not commute with the Hamiltonian and so does not constitute a constant of motion. Nevertheless, the component of $\mathbf{L}$ along $\hat{\mathbf{k}}$ is indeed a constant of motion. On the other hand, the separability property of the Schrödinger equation tells us that there exists another constant of motion which is related to the Runge-Lenz vector.

In Fig. 4, we plot the square modulus of $\psi_{n_1,n_2,m}(\mathbf{r})$ for the ground state as a function of $\mathbf{r}$, in the plane defined by $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$, and for $\lambda=1$. We can see that the effect of the repulsive parabolic barrier over the probability distribution is to remove it from the direction $\hat{\mathbf{k}}$. It does not have spherical symmetry as a consequence of the form of the potential and the $|\psi_{0,0,0}(\mathbf{r})|^2$ maximum is out of the origin.

In Fig. 5, we compare the probability distribution given by $|\psi_{2,0,1}(\mathbf{r})|^2$ with the solution of the hydrogen atom when the parabolic quantum numbers are $n_1=2$, $n_2=0$, and $m=1$. The presence of the parabolic barrier modifies $|\psi_{2,0,1}(\mathbf{r})|^2$, removing the probability from the $\hat{\mathbf{k}}$ axes. We see that $\psi_{n_1,n_2,m}(\mathbf{r})$ maintains the distribution asymmetry

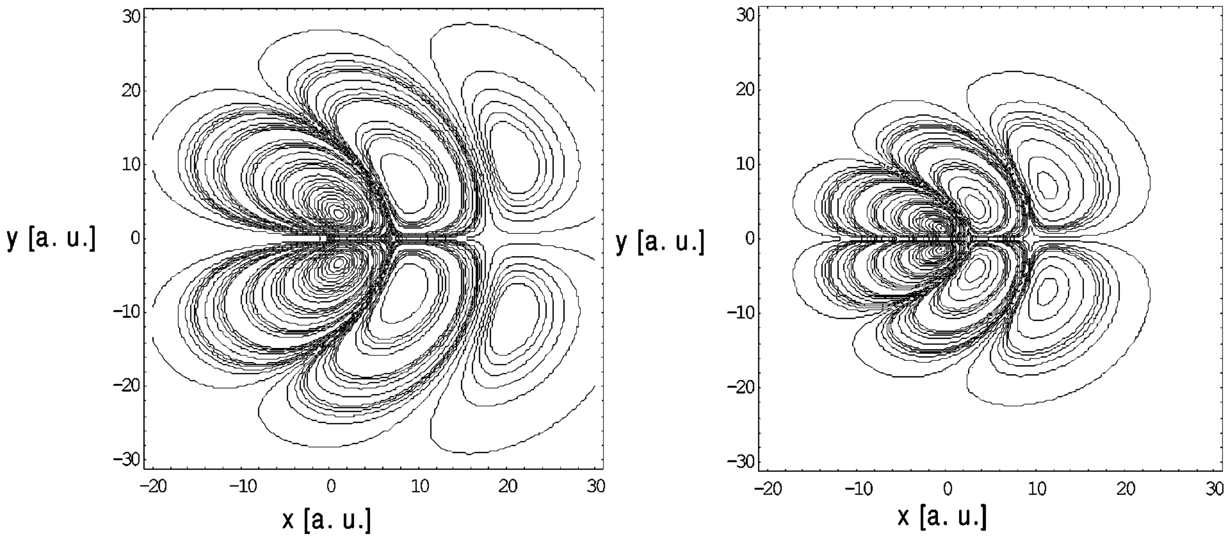

FIG. 5. Probability distribution given by $|\psi_{2,0,1}(\mathbf{r})|^2$ compared with the solution of the hydrogen atom for $n_1=2$, $n_2=0$, and $m=1$.

that can be seen in the hydrogen atom for certain combinations of the parabolic quantum numbers. For $n_1>n_2$, the probability of finding the particle in the direction $\hat{\mathbf{k}}\cdot\mathbf{r}>0$ is greater than for $\hat{\mathbf{k}}\cdot\mathbf{r}>0$, and vice versa for $n_2>n_1$. Besides, it should be noted that this asymmetry also depends on λ.

II. ASYMPTOTIC BEHAVIOR

In order to study the asymptotic behavior of the solutions given by Eqs. (14) and (16), we make use of the $r\rightarrow\infty$ form for ${}_1F_1[b,c,x]$:

$$ {}_1F_1[b,c,x]=\frac{\Gamma(c)}{\Gamma(c-b)}(-x)^{-b}v_1[b,b-c+1,-x] + \frac{\Gamma(c)}{\Gamma(b)}x^{b-c}e^{x}v_2[1-b,c-b,x], \qquad (21)$$

where $v_1[a,b,x]$ and $v_2[a,b,x]$ are the Whittaker functions [16]. Replacing Eq. (21) in Eq. (14) and to order $1/r$, we find

$$\psi_1(\mathbf{r})\rightarrow N\Bigg[\frac{(-1)^{\lambda}\Gamma(1+2\lambda)}{\Gamma(1+\lambda+i|Z|\mu/k)} \times e^{-(\pi/2)(|Z|\mu/k)}e^{i\mathbf{k}\cdot\mathbf{r}+i(|Z|\mu/k)\ln[kr+\mathbf{k}\cdot\mathbf{r}]} + \frac{\Gamma(1+2\lambda)(-i)^{-i|Z|\mu/k-1}}{\Gamma(-i|Z|\mu/k+\lambda)}\,\frac{e^{-ikr-i|Z|\mu/k\,\ln[kr+\mathbf{k}\cdot\mathbf{r}]}}{[kr+\mathbf{k}\cdot\mathbf{r}]}\Bigg].$$

Choosing an outgoing unitary flux normalization results in

$$\psi_1(\mathbf{r})\rightarrow e^{i\mathbf{k}\cdot\mathbf{r}+i(|Z|\mu/k)\ln[kr+\mathbf{k}\cdot\mathbf{r}]}+f_{\lambda}(\theta)\,\frac{e^{-ikr-i(|Z|\mu/k)\ln[2kr]}}{r}, \qquad (22)$$

where

$$N_{1,\lambda}=(-1)^{-\lambda}\,\frac{\Gamma(1+\lambda+i|Z|\mu/k)}{\Gamma(1+2\lambda)}\,e^{(\pi/2)(|Z|\mu/k)}, \qquad (23)$$

$$f_{1,\lambda}(\theta)=(-1)^{-\lambda}\Bigg(|Z|\mu\frac{e^{-i(|Z|\mu/k)\,\ln[\cos^2\theta/2]+2i\gamma_p}}{2k^2\cos^2\theta/2} + i\lambda\,\frac{e^{-i(|Z|\mu/k)\ln[\cos^2\theta/2]+2i\gamma_p}}{2k\cos^2\theta/2}\Bigg) = (-1)^{-\lambda}[f_{1,c}(\theta)+f_{1,p}(\theta)], \qquad (24)$$

and $\gamma_p=\arctan[|Z|\mu/k(1+\lambda)]$. Scattering amplitude $f_{1,\lambda}(\theta)$ turns out to be the addition of two different amplitudes $f_{1,c}(\theta)$ and $f_{1,p}(\theta)$. $f_{1,c}(\theta)$ is similar to the Coulomb scattering amplitude, the only difference being γ_p, and reduces exactly to it for $\lambda=0$. $f_{1,p}(\theta)$ shows the same angular behavior as $f_{1,c}(\theta)$, while its amplitude is proportional to λ and inversely proportional to k. It can be concluded that $f_{1,c}(\theta)$ and $f_{1,p}(\theta)$ are associated with the transition amplitudes for the Coulomb and the parabolic potential V^{+}, respectively.

The normalized solution ψ_1 reads

$$\psi_1(\mathbf{r})=N_{1,\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}[-i(kr+\mathbf{k}\cdot\mathbf{r})]^{\lambda} \times {}_1F_1\left[-i\frac{|Z|\mu}{k}+\lambda,1+2\lambda,-i(kr+\mathbf{k}\cdot\mathbf{r})\right]. \qquad (25)$$

In the following section we analyze the cross section defined by $f_{1,\lambda}(\theta)$. The distortion that accompanies the plane wave in Eq. (14) or Eq. (25) depends only on one of the parabolic coordinates, while that in ψ_2, given by Eq. (16), depends on both of them.

Replacing Eq. (21) in Eq. (16) and retaining terms to order $1/r$, we obtain

$$\psi_2(\mathbf{r})\rightarrow e^{i\mathbf{k}\cdot\mathbf{r}-i(|Z|\mu/k)\ln[kr-\mathbf{k}\cdot\mathbf{r}]}+f_{2,\lambda}(\theta)\frac{e^{ikr+i(|Z|\mu/k)\ln[2kr]}}{r} + g_{2,\lambda}(\theta)\,\frac{e^{-ikr-i(|Z|\mu/k)\ln[2kr]}}{r}, \qquad (26)$$

where

$$N_2=(-1)^{\lambda}\frac{\Gamma(1+2\lambda)}{\Gamma(1+\lambda)}\Gamma(1-i|Z|\mu/k)e^{-(\pi/2)(|Z|\mu/k)}, \tag{27}$$

$$f_{2,\lambda}(\theta)=f_c^{+}(\theta)=|Z|\mu\frac{\Gamma(1+i|Z|\mu/k)}{\Gamma(1-i|Z|\mu/k)}\frac{e^{-i(|Z|\mu/k)\ln[\sin^2\theta/2]+2i\gamma_c}}{2k^2\sin^2\theta/2}, \tag{28}$$

$$g_{2,\lambda}(\theta)=i(-1)^{-\lambda}\lambda\frac{e^{-i(|Z|\mu/k)}\ln[\sin^2\theta/2]}{2k\cos^2\theta/2}, \tag{29}$$

and $\gamma_c=\arctan[|Z|\mu/k]$. $f_c^{+}(\theta)$ represents the Coulomb transition amplitude associated with the outgoing wave function.

It can be seen from Eq. (26) that the function ψ_2 in the asymptotic region is a sum of three terms. The first one shows the well known eikonal behavior corresponding to the nonperturbed Coulomb potential. The other two terms represent outgoing and incoming spherical wave functions modulated by their corresponding scattering amplitudes. The eikonal wave arises from the asymptotic behavior of the hypergeometric function that depends on η in Eq. (16) since the first order of the Kummer function in ξ is constant. It is also clear that the outgoing spherical wave contributes to the outgoing flux, which is driven only by the distorted plane wave in the pure Coulomb case, and hence $\psi_2(\mathbf{r})$ has been normalized to unit outgoing flux.

When the Coulomb potential is absent, i.e., $|Z|=0$, the wave function corresponding to V^{+} is given by Eq. (17). The asymptotic behavior for $\psi_3(\mathbf{r})$ can be shown by replacing Eq. (21) in Eq. (17):

$$\psi_3(\mathbf{r})\rightarrow e^{i\mathbf{k}\cdot\mathbf{r}}+f_{3,\lambda}(\theta)\frac{e^{-ikr}}{r}, \tag{30}$$

where

$$N_{3,\lambda}=(-1)^{-\lambda}\frac{\Gamma(1+\lambda)}{\Gamma(1+2\lambda)}, \tag{31}$$

$$f_{3,\lambda}(\theta)=i(-1)^{-\lambda}\frac{\lambda}{2k\cos^2\theta/2}. \tag{32}$$

The normalization coefficient is calculated in a way that assures unit outgoing flux:

$$\psi_3(\mathbf{r})=N_{3,\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}(-ik\xi)^{\lambda}{}_1F_1[\lambda,1+2\lambda,-ik\xi]. \tag{33}$$

Even when V^p is noncentral, the asymptotic behavior for $\psi_3(\mathbf{r})$ is not distorted, so we can consider V^p to be a short-range potential. The transition amplitude $f_{3,\lambda}(\theta)$ has an angular dependence quite similar to that for the Coulomb amplitude. However, its dependence on energy is quite different.

In the following section, we will study the scattering cross sections for $V^{\pm}$.

III. DIFFERENTIAL CROSS SECTION

The scattering cross section $\sigma(\theta)$ for any given potential is defined from the asymptotic form for the continuum wave function. As we are dealing with a long-range potential, a true free state is not possible [13]. Therefore, the asymptotic wave function includes a logarithmic phase which modifies both the plane and scattered wave:

$$\psi\rightarrow e^{i\mathbf{k}\cdot\mathbf{r}+ia\ln[kr+\mathbf{k}\cdot\mathbf{r}]}+f^{\pm}(\theta)\frac{e^{\pm ikr-ia\ln[2kr]}}{r}, \tag{34}$$

where $f^{\pm}(\theta)$ is the scattering amplitude. a is a parameter that depends in general on the ratio between the intensity of the long-range portion of the given potential and the asymptotic particle velocity. In our case, this is simply $a=Z\mu/k$. As usual, the cross section can be written in terms of the scattering amplitude:

$$\sigma^{\pm}(\theta)=|f^{\pm}(\theta)|^2. \tag{35}$$

Since the potential $V^{\pm}$ is defined as the sum of two different terms, the corresponding cross section can be cast into

$$\sigma_{\lambda}^{-}(\theta)=\sigma_c(\theta)+\sigma_{\lambda}^{p+}(\theta)+\sigma_{\text{int}}^{-}(\theta), \tag{36}$$

where

$$\sigma_c(\theta)=\frac{(|Z|\mu)^2}{4k^4\cos^4\theta/2}, \tag{37}$$

$$\sigma_{\lambda}^{p+}(\theta)=\frac{\lambda^2}{4k^2\cos^4\theta/2}, \tag{38}$$

$$\sigma_{\text{int}}^{-}(\theta)=2\,\text{Re}[f_c^1(\theta)f_p^1(\theta)], \tag{39}$$

i.e., the cross section can be written as the sum of three terms, one associated to the scattering off the Coulomb potential, the second associated to the scattering off the parabolic potential V^p, and the last one corresponding to the interference between the first two amplitudes.

It can be shown that the angular dependence of $\sigma_{\lambda}^{-}(\theta)$ is similar to the Rutherford scattering. In the low energy limit, $k\rightarrow 0$, $\sigma_c(\theta)$ dominates the cross section, giving a pure Rutherford cross section. On the other hand, when $k\rightarrow\infty$, the incident particle interacts with V^p in a stronger way, thus making $\sigma_{\lambda}^{p+}(\theta)$ dominate the differential cross section.

Total cross section defined as the angular integration of $\sigma_{\lambda}^{-}(\theta)$ is divergent. This is a well known feature of long-range potentials such as the Coulomb potential.

IV. DISCUSSION AND OUTLOOK

We have found closed-form solutions for the continuum as well as the discrete spectra for a noncentral potential built from a Coulomb potential plus a parabolic barrier.

We have analyzed the probability distribution obtained from these solutions. Similarities and differences from the pure Coulomb problem are highlighted. The presence of the parabolic barrier destroys the system's, spherical symmetry, but still it is possible to find three constants of motion related to the energy, the angular momentum projection along the $\hat{\mathbf{k}}$

direction, and the Runge-Lenz vector, respectively.

These two-body wave functions are related to the three-body ones. So it could be expected that conclusions drawn for them could be extended, at least qualitatively, to the three-body problem.

In the three-body continuum Coulomb problem, it has been customary to write the Schrödinger equation in a set of six parabolic coordinates [5,11,12,17]. This set naturally leads to the outgoing (or incoming) asymptotic behavior associated with each of the parabolic coordinates [18,19,17]. For example, the asymptotic motion of three particles moving away from each other is represented by a plane wave distorted by three eikonal phases that depend on three parabolic coordinates and are independent of the other three. The only way in which outgoing (incoming) solutions have been obtained so far was considering that the total wave function is constrained to depend on three parabolic coordinates leading to outgoing (incoming) behavior [11,12,17].

However, we note that the ψ_2 function [Eq. (16)] has a purely outgoing asymptotic behavior, but retains its dependence on all the parabolic coordinates (in our case, ξ and η) for nonasymptotic regions.

This fact implies a change in the normalization factor with respect to the Coulomb problem. Then, we can think of three-body wave functions depending on all the six coordinates, but with the right asymptotic behavior given by the eikonal functions in only three of them. In this way, we would be able to introduce modifications in the normalization factors. Moreover, one can expect these kinds of wave functions to describe a richer dynamics (particularly in the so-called condensation region) by retaining the full dependence on the system's six degrees of freedom.

[1] D.R. Bates and R. McCarroll, Proc. R. Soc. London, Ser. A **245**, 175 (1958).

[2] C.D. Lin, Phys. Rep. **257**, 1 (1995).

[3] S.Y. Ovchinnikov, J. Macek, and D.B. Khrebtukov, Phys. Rev. A **56**, 2872 (1997).

[4] C.R. Garibotti and J.E. Miraglia, Phys. Rev. A **21**, 572 (1980); **25**, 1440 (1982).

[5] F.D. Colavecchia, G. Gasaneo, and C.R. Garibotti, Phys. Rev. A **57**, 1018 (1998).

[6] J. Berakdar and J.S. Briggs, Phys. Rev. Lett. **72**, 3799 (1994); J. Phys. B **27**, 4271 (1994).

[7] J. Berakdar, Phys. Rev. A **53**, 2314 (1996).

[8] B.A. Lippmann, Phys. Rev. **102**, 264 (1956).

[9] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Davey, New York, 1965), Vol. 69.

[10] L.D. Dodd and K.R. Graider, Phys. Rev. **146**, 675 (1966); see also K.R. Graider and L.D. Dodd, *ibid.* **146**, 671 (1966).

[11] G. Gasaneo, F.D. Colavecchia, C.R. Garibotti, J.E. Miraglia, and P. Macri, Phys. Rev. A **55**, 2809 (1997); P. Macri, J.E. Miraglia, C.R. Garibotti, F.D. Colavecchia, and G. Gasaneo, *ibid.* **55**, 3518 (1997).

[12] G. Gasaneo, Ph.D. thesis, Instituto Balseiro, 1997 (unpublished).

[13] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, New York, 1965).

[14] H.A. Bethe and E.E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Springer-Verlag, Berlin, 1957).

[15] L.J. Slater, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover Publications Inc., New York, 1970), p. 505.

[16] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. I.

[17] H. Klar, Z. Phys. D **16**, 231 (1990).

[18] M.R.H. Rudge and M.J. Seaton, Proc. R. Soc. London, Ser. A **283**, 262 (1965); Rev. Mod. Phys. **40**, 564 (1968).

[19] R. Peterkops, Phys. Lett. **62A**, 81 (1977); *Theory of Ionization of Atoms by Electron Impact* (Colorado Associated Press, Boulder, 1977); J. Phys. B **15**, L751 (1981).