# Hypergeometric integrals arising in atomic collisions physics

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(Received 16 January 1997; accepted for publication 25 June 1997)

We introduce a method of obtaining volume integrals involving confluent hypergeometric functions. This method is based on the integral representation of these functions and enabled us to write a generalized Nordsieck integral in terms of hypergeometric functions of many variables. We explore some particular results that could be useful when calculating transition matrices in collision theories. © *1997 American Institute of Physics.* [S0022-2488(97)01312-1]

## I. INTRODUCTION

Integrals involving confluent hypergeometric functions are spread through a variety of topics in physics. Most of these integrals appear in the field of atomic collisions theories. It is well known that the solution of a system of two charged particles is analytically described using a confluent Kummer function ${}_1F_1(a,c,x)$. For example, the computation of transition matrices of ionization processes or normalization factors of wave functions involve the calculation of integrals with two Kummer functions. Three body processes such as electron or ion-atom collisions would include more complicated forms of these integrals.

The first results concerning the integrals with hypergeometric functions are related to the calculation of transition matrices in the First Born Approximation.[1] In this case, the integral looks like

$$J^{FBA}=\int \frac{d\mathbf{r}}{r} e^{-zr+i\mathbf{q}.\mathbf{r}} \; {}_1F_1(ia_1,1,ip_1r+i\mathbf{p}_1.\mathbf{r}), \tag{1}$$

where the exponential decreasing factor describes the bounded 1s state of an hydrogenic atom, the inverse function represents the Coulomb potential and the confluent hypergeometric is the final continuum state of the ionized electron. The oscillatory exponential arises from the plane waves of the initial and final states of the projectile. This integral can be solved by using a contour representation for the Kummer function.[2]

A natural extension of (1) arises when dealing with the bremsstrahlung of heavy elements at relativistic high energies

$$J_1=\int \frac{d\mathbf{r}}{r} e^{-zr+i\mathbf{q}.\mathbf{r}} \mathscr{F}_1\mathscr{F}_2 \tag{2}$$

with $\mathscr{F}_j={}_1F_1(ia_j,1,ip_jr+i\mathbf{p}_j.\mathbf{r})$. The first calculation of such integral has been performed by Bess.[3] He expressed the ${}_1F_1(a,1,x)$ function through an integral representation in terms of Bessel $J_0(x)$ and extensively used properties of all Bessel functions. Although Bess solved completely the problem and gave an analytic expression of (2); Nordsieck[4] retrieved his result from an elegant and concise calculation using a contour representation of ${}_1F_1(a,1,x)$ and the wide use of his method finally named these type of integrals.

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This work is motivated from a recent proposal of Gasaneo *et al.*[5] They obtained an approximate solution of the three body continuum Coulomb problem in terms of the two variable hypergeometric function $\Phi_2$. This function has a series expression in terms of products of two Kummer functions. In order to be able to obtain transition matrices of an ionization process, it is necessary to extend the Bess and Nordsieck results to the cases where the second parameter of the confluent hypergeometric functions of (2) is no longer 1, but any real positive number. Our result is not restricted to this case and many other applications of the following formulas can be found. For example, it allows to obtain transition matrices of excitation or ionization processes from excited states of hydrogenic atoms in a natural way.

The plan of the paper is as follows. In section II we briefly discuss the methodology and present the main result in some useful ways. In section III we obtain some restricted results, provide the guidelines for the calculation of some other Nordsieck-like integrals, and summarize our work. The Appendix reviews the definition of multiple variable hypergeometric functions thoroughly used.

## II. THE GENERALIZED NORDSIECK'S INTEGRAL $J_1'$

We are interested in obtaining an analytical expression for the integral

$$J_1' = \int \frac{d\mathbf{r}}{r} e^{-zr+i\mathbf{q}.\mathbf{r}} \mathscr{F}_1' \mathscr{F}_2' \tag{3}$$

with $\text{Re}(z)>0$ and $\mathscr{F}_j' = {}_1F_1(ia_j, b_j, ip_j r + i\mathbf{p}_j.\mathbf{r})$. The Nordsieck's integral $J_1$ is defined as[6]

$$J_1 = J_1'(b_1 = b_2 = 1).$$

As a general notation, the primed integrals or functions will include the parameters $b_j \neq 1$. As we pointed out before, the methods used to solve the integral $J_1$ cannot be easily extended to treat the general case $b_1 \neq b_2 \neq 1$. Instead of previous works, we consider here the integral representation of the confluent hypergeometric function

$${}_1F_1(a,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xu} u^{a-1} (1-u)^{c-a-1}\, du \tag{4}$$

with $\text{Re}(c) > \text{Re}(a) > 0$.[7] These conditions limits the election of the parameters $b_i$. Besides, in order to properly use this expression, we should include a real positive integrating factor $\varepsilon$ in the first parameters of the hypergeometric functions in (3), that is to say we should replace $ia_i \to ia_i + \varepsilon$, and take the limit $\varepsilon \to 0^+$ after solving the integral. However, for the sake of simplicity, we will not include this parameter explicitly in the next formulas. We note that using the integral representations, the triply integral (3) transforms into a five dimensional integral

$$J_1' = \int_0^1 u^{ia_1 - 1}(1-u)^{b_1 - ia_1 - 1}\, du \int_0^1 v^{ia_2-1}(1-v)^{b_2 - ia_2 - 1}\, dv \tag{5}$$

$$\times \int \frac{d\mathbf{r}}{r} e^{-zr+i\mathbf{q}.\mathbf{r}} \quad e^{iu(p_1 r + \mathbf{p}_1.\mathbf{r})} e^{iv(p_2 r + \mathbf{p}_2.\mathbf{r})}.$$

However, this procedure converts the hypergeometrics $\mathscr{F}_j'$ into simple exponential functions and the spatial integral is easily solved

$$I = \int \frac{d\mathbf{r}}{r} e^{-zr+i\mathbf{q}.\mathbf{r}} e^{iu(p_1 r + \mathbf{p}_1.\mathbf{r})} e^{iv(p_2 r + \mathbf{p}_2.\mathbf{r})} = \int \frac{d\mathbf{r}}{r} e^{-\lambda r + i\mathbf{Q}.\mathbf{r}} = \frac{4\pi}{\lambda^2 + \mathbf{Q}^2}, \tag{6}$$

where we have defined

$$\lambda = z - iup_1 - ivp_2 \quad \mathbf{Q} = \mathbf{q} + u\mathbf{p}_1 + v\mathbf{p}_2 . \tag{7}$$

Following Gravielle and Miraglia,[6] we introduce the shorthand notation

$$D = z^2 + \mathbf{q}^2, \tag{8}$$

$$S_i = \mathbf{q}.\mathbf{p}_i - izp_i \quad i = 1,2, \tag{9}$$

$$S_3 = p_1 p_2 - \mathbf{p}_1.\mathbf{p}_2 , \tag{10}$$

and

$$U_i = \frac{2S_i}{D}, \quad A_i = 1 + U_i \quad \text{with } i = 1,2,3. \tag{11}$$

Then we have

$$I = \frac{4\pi}{D}[1 + uU_1 + vU_2 + uvU_3]^{-1}. \tag{12}$$

The key step in the present method is to re-write the last expression in such a way that the remaining two integrals resembles a representation of a multiple variable hypergeometric function. After some algebra we find

$$I = \frac{4\pi}{DA_1A_2}\left[\left(\frac{1-u}{A_1} + u\right)\left(\frac{1-v}{A_2} + v\right) - uvx_0\right]^{-1} \tag{13}$$

with

$$x_0 = 1 - \frac{A_1 + A_2 - A_3}{A_1A_2}. \tag{14}$$

The last expression for $I$ suggest the following change of variables:

$$s = \frac{A_1 u}{1 + uU_1} , \quad t = \frac{A_2 v}{1 + vU_2} , \tag{15}$$

which conserves the limits of the integral. Then we obtain the following expression for $J_1'$:

$$J_1' = 4\pi \frac{\Gamma(b_1)}{\Gamma(ia_1)\Gamma(b_1 - ia_1)} \frac{\Gamma(b_2)}{\Gamma(ia_2)\Gamma(b_2 - ia_2)} \frac{A_1^{-ia_1}A_2^{-ia_2}}{D} \times \int_0^1 ds \int_0^1 dt\ s^{ia_1-1}t^{ia_2-1}(1-s)^{b_1-ia_1-1}(1-t)^{b_2-ia_2-1} \times (1-sz_1)^{1-b_1}(1-tz_2)^{1-b_2}(1-stx_0)^{-1}, \tag{16}$$

where

$$z_i = \frac{U_i}{A_i}. \tag{17}$$

The double integral in (16) corresponds to a representation of a three variable hypergeometric function,[8] but, for the sake of clarity, we go a step further to obtain the corresponding series representation. Taking into account the series behavior

$$(1-ax)^{-q}=\sum_{n=0}^{\infty}(q)_n\frac{(ax)^n}{n!}\quad \text{with} \quad |ax|<1, \tag{18}$$

we can expand the factors which include the variables $\{x_0,z_1,z_2\}$ using this formula to get

$$\begin{aligned}J_1'={}&4\pi\frac{\Gamma(b_1)}{\Gamma(ia_1)\Gamma(b_1-ia_1)}\frac{\Gamma(b_2)}{\Gamma(ia_2)\Gamma(b_2-ia_2)}\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\\&\times\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(b_1-1)_l(b_2-1)_m(1)_n\frac{z_1^l}{l!}\frac{z_2^m}{m!}\frac{x_0^n}{n!}\\&\times\int_0^1 s^{ia_1-1+l+n}(1-s)^{b_1-ia_1-1}ds\int_0^1 t^{ia_2-1+n+m}(1-t)^{b_2-ia_2-1}\,dt.\end{aligned} \tag{19}$$

The remaining integrals can be associated with the confluent hypergeometric ${}_1F_1$ evaluated in zero:

$$\int_0^1 e^{xu}u^{a-1}(1-u)^{c-a-1}\,du={}_1F_1(a,c,0)=\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \tag{20}$$

Then we obtain a series representation for $J_1'$ that can be identified with the three variable hypergeometric function $F^{(3)}$

$$J_1'=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(ia_1)_{l+n}(ia_2)_{m+n}(b_1-1)_l(b_2-1)_m(1)_n}{(b_1)_{l+n}(b_2)_{m+n}}\frac{z_1^l}{l!}\frac{z_2^m}{m!}\frac{x_0^n}{n!}, \tag{21}$$

$$=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}F^{(3)}\left[\begin{matrix}-;ia_1,ia_2,-;b_1-1,b_2-1,1\\ -;b_1,b_2,-;-\end{matrix}\middle| z_1,z_2,x_0\right], \tag{22}$$

where we have used the notation of Srivastava and Manocha[8] for the generalized hypergeometric function of three variables (see the Appendix). Expression (22) is the main result of this work and the most compact form of $J_1'$, but since the properties of these generalized hypergeometric functions are not well known, we will search some other useful expressions.

Instead of using the series form (18) for all variables, we can expand only $(1-stx_0)^{-1}$

$$\begin{aligned}J_1'={}&4\pi\frac{\Gamma(b_1)}{\Gamma(ia_1)\Gamma(b_1-ia_1)}\frac{\Gamma(b_2)}{\Gamma(ia_2)\Gamma(b_2-ia_2)}\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{n=0}^{\infty}(1)_n\frac{x_0^n}{n!}\\&\times\int_0^1 ds\; s^{ia_1+n-1}(1-s)^{b_1-ia_1-1}(1-sz_1)^{1-b_1}\\&\times\int_0^1 dt\; t^{ia_2+n-1}(1-t)^{b_2-ia_2-1}(1-tz_2)^{1-b_2},\end{aligned} \tag{23}$$

where we immediately recognize the integral representation of hypergeometric function ${}_2F_1$

$$I_i=\int_0^1 ds\ s^{ia_i+n-1}(1-s)^{b_i-ia_i-1}(1-sz_1)^{1-b_i}$$
$$=\frac{\Gamma(ia_i+n)\Gamma(b_i-ia_i)}{\Gamma(b_i+n)}\ {}_2F_1(b_i-1,ia_i+n,b_i+n,z_i) \tag{24}$$

and then

$$J_1'=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{n=0}^{\infty}\frac{(ia_1)_n(ia_2)_n(1)_n}{(b_1)_n(b_2)_n}\frac{x_0^n}{n!}$$
$$\times{}_2F_1(b_1-1,ia_1+n,b_1+n,z_1)\ \ {}_2F_1(b_2-1,ia_2+n,b_2+n,z_2). \tag{25}$$

Finally, the last expression we will give here corresponds to the expansion of the uncoupled kernels that include the variables $z_i$

$$J_1'=4\pi\frac{\Gamma(b_1)}{\Gamma(ia_1)\Gamma(b_1-ia_1)}\frac{\Gamma(b_2)}{\Gamma(ia_2)\Gamma(b_2-ia_2)}\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}$$
$$\times\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(b_1-1)_l(b_2-1)_m\frac{z_1^l}{l!}\frac{z_2^m}{m!}$$
$$\times\int_0^1\int_0^1 ds\ dt\ s^{ia_1-1+l}(1-s)^{b_1-ia_1-1}t^{ia_2-1+m}(1-t)^{b_2-ia_2-1}(1-stx_0)^{-1}. \tag{26}$$

In this case the double integral can be identified with the one variable generalized hypergeometric function ${}_3F_2$[7]

$$I=\int_0^1\int_0^1 ds\ dt\ s^{ia_1-1+l}(1-s)^{b_1-ia_1-1}t^{ia_2-1+m}(1-t)^{b_2-ia_2-1}(1-stx_0)^{-1}$$
$$=\frac{\Gamma(ia_1+l)\Gamma(b_1-ia_1)}{\Gamma(b_1+l)}\frac{\Gamma(ia_2+m)\Gamma(b_2-ia_2)}{\Gamma(b_2+m)}$$
$$\times{}_3F_2(1,ia_1+l,ia_2+m,b_1+l,b_2+m,x_0), \tag{27}$$

and then

$$J_1'=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\frac{(b_1-1)_l(b_2-1)_m(ia_1)_l(ia_2)_m}{(b_1)_l(b_2)_m}\frac{z_1^l}{l!}\frac{z_2^m}{m!}$$
$$\times{}_3F_2(1,ia_1+l,ia_2+m,b_1+l,b_2+m,x_0). \tag{28}$$

We note that Eqs. (25) and (28) can be viewed as different series representations of function $F^{(3)}$ for the given set of parameters.

## III. SOME PARTICULAR CASES

Now we will study several interesting cases of the generalized Nordsieck integral. We will follow the notation of Gravielle and Miraglia.[6]

### A. Nordsieck integrals

In order to check our results, we restrict the parameters to the case $b_1=b_2=1$ and hence we obtain the integral

$$J_1=4\pi\frac{1}{\Gamma(ia_1)\Gamma(1-ia_1)}\frac{1}{\Gamma(ia_2)\Gamma(1-ia_2)}\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\times\int_0^1\int_0^1 ds\; dt\; s^{ia_1-1}(1-s)^{-ia_1}t^{ia_2-1}(1-t)^{-ia_2}(1-stx_0)^{-1}=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\;{}_2F_1(ia_1,ia_2,1,x_0), \tag{29}$$

where we have used the formula (A5) presented in the Appendix. This is the restricted result obtained by Bess[3] and Nordsieck.[4] A second useful expression can be given when either $b_1=1$ or $b_2=1$. For the sake of brevity we choose $b_1=1$. From Eq. (25) we can see that since ${}_2F_1(0,ia_1+n,1+n,z_1)=1$

$$J_1'(b_1=1,b_2\neq 1)=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{n=0}^{\infty}\frac{(ia_1)_n(ia_2)_n}{(b_2)_n}\frac{x_0^n}{n!}\;{}_2F_1(b_2-1,ia_2+n,b_2+n,z_2). \tag{30}$$

After expanding the ${}_2F_1$ function in power series we can write this integral in terms of two variables Appell hypergeometric function[9]

$$J_1'(b_1=1,b_2\neq 1)=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b_2-1)_m(ia_1)_n(ia_2)_{m+n}}{(b_2)_{m+n}}\frac{z_2^m}{m!}\frac{x_0^n}{n!}=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D}F_1(ia_2,b_2-1,ia_1,b_2,z_2,x_0). \tag{31}$$

Another useful integral is

$$J_2'=\int d\mathbf{r}e^{-zr+i\mathbf{q}.\mathbf{r}}\mathscr{F}_1'\mathscr{F}_2'=-\frac{\partial J_1'}{\partial z}=4\pi\frac{A_1^{-ia_1}A_2^{-ia_2}}{D^2}\left\{J_{21}F^{(3)}[z_1,z_2,x_0]+\frac{1}{A_1A_2}\left[\frac{ia_1ia_2}{b_1b_2}J_{22}^0F^{(3)}[z_1,z_2,x_0^+]+\frac{(b_1-1)ia_1}{b_1}J_{22}^1F^{(3)}[z_1^+,z_2,x_0]+\frac{(b_2-1)ia_2}{b_2}J_{22}^2F^{(3)}[z_1,z_2^+,x_0]\right]\right\} \tag{32}$$

with

$$J_{21}=2z-\frac{ia_1B_1}{A_1}-\frac{ia_2B_2}{A_2}, \tag{33}$$

$$J_{22}^0=\frac{(A_2-A_3)}{A_1}B_1+\frac{(A_1-A_3)}{A_2}B_2+B_3, \tag{34}$$

$$J_{22}^{1}=\frac{A_2B_1}{A_1},\quad J_{22}^{2}=\frac{A_1B_2}{A_2}, \tag{35}$$

$$B_i=2(ip_j+zU_j),\quad B_3=2zU_3, \tag{36}$$

and

$$F^{(3)}[z_1^+,z_2,x_0]=F^{(3)}\left[\begin{matrix}-;ia_1+1,ia_2,-;b_1,b_2-1,1\\ -;b_1+1,b_2,-;-\end{matrix}\middle| z_1,z_2,x_0\right], \tag{37}$$

$$F^{(3)}[z_1,z_2^+,x_0]=F^{(3)}\left[\begin{matrix}-;ia_1,ia_2+1,-;b_1-1,b_2,1\\ -;b_1,b_2+1,-;-\end{matrix}\middle| z_1,z_2,x_0\right], \tag{38}$$

$$F^{(3)}[z_1,z_2,x_0^+]=F^{(3)}\left[\begin{matrix}-;ia_1+1,ia_2+1,-;b_1-1,b_2-1,2\\ -;b_1+1,b_2+1,-;-\end{matrix}\middle| z_1,z_2,x_0\right], \tag{39}$$

represent the partial derivatives of the function $F^{(3)}[z_1,z_2,x_0]$.

The integrals which involves gradients of hypergeometric functions appear in the calculation of radiative processes and can be derived from

$$J_3'=\int d\mathbf{r}\; e^{-zr+i\mathbf{q}.\mathbf{r}}\mathscr{F}_1'\nabla_{\mathbf{r}}.\mathscr{F}_2'=p_2\nabla_{p_2}J_1'. \tag{40}$$

### B. Integrals with eikonal phases

We can derive in a similar way the integrals involving eikonal phases. We define

$$\begin{aligned}\mathscr{E}_i'&=\gamma_i'\lim_{p_i\to\infty}{}_1F_1(ia_i,b_i,ip_ir+i\mathbf{p}_i.\mathbf{r})\\ &=\exp[-ia_i\ \ln(p_ir+\mathbf{p}_i.\mathbf{r})]\end{aligned} \tag{41}$$

with $\gamma_i'=\exp(\pi a_i/2)\Gamma(b_i-ia_i)/\Gamma(b_i)$. Then it is easy to see that the eikonal phases do not depend on the specific values of the constants $b_i$, that is to say $\mathscr{E}_i'=\mathscr{E}_i$ and then integrals such as

$$J_4'=\int\frac{d\mathbf{r}}{r}e^{-zr+i\mathbf{q}.\mathbf{r}}\mathscr{E}_1'\mathscr{E}_2' \tag{42}$$

are the same as those obtained by Reinhold and Miraglia.[10] For completeness we write their result

$$J_4'=J_4=\gamma_1'\gamma_2'\lim_{p_1,p_2\to\infty}J_1'=4\pi\gamma_1\gamma_2\frac{U_1^{-ia_1}U_2^{-ia_2}}{D}{}_2F_1(ia_1,ia_1,1,x_3) \tag{43}$$

with $x_3=1+U_3/(U_1U_2)$ and as usual $\gamma_i=\gamma_i'\ (b_i=1)$. Similar procedures to those applied in the context of $J_1'$, lead to the integrals

$$J_5'=J_5=\int d\mathbf{r}\; e^{-zr+i\mathbf{q}.\mathbf{r}}\mathscr{E}_1'\mathscr{E}_2'=-\frac{\partial J_4}{\partial z}, \tag{44}$$

$$J_6'=J_6=\int d\mathbf{r}\; e^{-zr+i\mathbf{q}.\mathbf{r}}\mathscr{E}_1'\nabla_{\mathbf{r}}\mathscr{E}_2'=p_2\nabla_{\mathbf{p}_2}J_4. \tag{45}$$

The situation is slightly different when dealing with integrals like

$$J_7' = \int d\mathbf{r}\; e^{-zr+i\mathbf{q}.\mathbf{r}} \mathscr{E}_2' \mathscr{F}_1' = \gamma_2' \lim_{p_2\to\infty} J_2' \tag{46}$$

because the hypergeometric $\mathscr{F}_1'$ has in fact $b_1 \neq 1$. After taking the limit we have

$$J_7' = 4\pi\gamma_2' \frac{A_1^{-ia_1} U_2^{-ia_2}}{D^2} \Bigg\{ J_{71} F^{(3)}[z_1,1,x_2] + \frac{1}{A_1 U_2} \Bigg[ \frac{ia_1 ia_2}{b_1 b_2} J_{72}^0 F^{(3)}[z_1,1,x_2^+] + \frac{(b_1-1)ia_1}{b_1} J_{72}^1 F^{(3)}[z_1^+,1,x_2] \Bigg] \Bigg\} \tag{47}$$

and

$$x_2 = \lim_{p_2\to\infty} x_0 = 1 - \frac{U_2 - U_3}{A_1 U_2}, \tag{48}$$

$$J_{71} = 2z - \frac{ia_1 B_1}{A_1} - \frac{ia_2 B_2}{U_2}, \tag{49}$$

$$J_{72}^0 = \frac{(U_2 - A_3)}{A_1} B_1 + \frac{(A_1 - A_3)}{U_2} B_2 + B_3, \tag{50}$$

$$J_{72}^1 = \frac{U_2 B_1}{A_1}. \tag{51}$$

The functions $F^{(3)}[z_1,1,x_2]$ can be reduced to a two variable $F_1$ function[9]

$$F^{(3)}[z_1,1,x_2] = \frac{\Gamma(b_2)\Gamma(1-ia_2)}{\Gamma(b_2 - ia_2)} F_1(ia_1, b_1-1, ia_2, b_1, z_1, x_2). \tag{52}$$

This result can be used to evaluate the derivatives of $F^{(3)}[z_1,1,x_2]$ appearing in (47). From the set of integrals $J_1' \cdots J_7'$ many other integrals can be obtained in similar ways.

### C. Integrals involving only one hypergeometric function

Finally, we show a particular result with one function $\mathscr{F}_1'$. These integrals can be reduced from the $J_1'$ by

$$L_n' = \int d\mathbf{r}\; e^{-zr+i\mathbf{q}.\mathbf{r}} r^n \mathscr{F}_1' = (-1)^{n+1} \frac{\partial^{n+1}}{\partial z^{n+1}} J_1'(a_2=0). \tag{53}$$

We present the following expressions useful for the evaluation of integrals similar to those obtained by Belkic[11,12]

$$L_{-1}' = J_1'(a_2=0) = 4\pi \frac{A_1^{-ia_1}}{D}\; {}_2F_1(b_1-1, ia_1, b_1, z_1) \tag{54}$$

$$L_0=J_2'(a_2=0)=-\frac{\partial J_1'(a_2=0)}{\partial z}$$
$$=\frac{4\pi A_1^{-ia_1}}{D^2}\left[L_{01}\ {}_2F_1(b_1-1,ia_1,b_1,z_1)+\frac{(b_1-1)ia_1}{b_1}\right.$$
$$\left.\times L_{02}^1\ {}_2F_1(b_1,ia_1+1,b_1+1,z_1)\right] \tag{55}$$

with

$$L_{01}=2z-\frac{ia_1B_1}{A_1},\quad L_{02}^1=\frac{B_1}{A_1^2}.$$

In summary, we have obtained a closed form for a set of hypergeometric integrals. We used an integral representation of the confluent hypergeometric functions that allow us to replace the $\mathscr{F}_j'$ functions by simple exponentials, but introducing two more integrals in the calculation. We make use of multiple variables hypergeometric functions to get analytic compact forms of these integrals. We have treated some particular extensions, such as some Nordsieck integrals; and described the way to obtain other results that have not been included here. Further research is been carried out in order to extend this work to six dimensional integrals which involves three or more Kummer hypergeometric functions that also appear in atomic theories.

## ACKNOWLEDGMENT

We would like to acknowledge J. E. Miraglia for helpful discussions.

## APPENDIX: DEFINITION OF THE HYPERGEOMETRIC FUNCTION $F^{(3)}$

Following Srivastava and Manocha,[8] we define here the three variables hypergeometric function

$$F^{(3)}\left[\begin{matrix}(\alpha);(\beta),(\beta'),(\beta'');(\gamma),(\gamma'),(\gamma'')\\(\varepsilon);(\zeta),(\zeta'),(\zeta'');(\delta),(\delta'),(\delta'')\end{matrix}\middle| x,y,z\right]=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Lambda(l,m,n)\frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}, \tag{A1}$$

where $(\alpha)$ abbreviates the array of A parameters $\alpha_1,\alpha_2,\ldots,\alpha_A$, etc. and

$$\Lambda(l,m,n)=\frac{\prod_{j=1}^{A}(\alpha_j)_{l+m+n}\prod_{j=1}^{B}(\beta_j)_{l+m}\prod_{j=1}^{B'}(\beta_j')_{m+n}\prod_{j=1}^{B''}(\beta_j'')_{l+n}}{\prod_{j=1}^{E}(\varepsilon_j)_{l+m+n}\prod_{j=1}^{G}(\zeta_j)_{l+m}\prod_{j=1}^{G'}(\zeta_j')_{m+n}\prod_{j=1}^{G''}(\zeta_j'')_{l+n}}$$
$$\times\frac{\prod_{j=1}^{C}(\gamma_j)_l\prod_{j=1}^{C'}(\gamma_j')_m\prod_{j=1}^{C''}(\gamma_j'')_n}{\prod_{j=1}^{D}(\delta_j)_l\prod_{j=1}^{D'}(\delta_j')_m\prod_{j=1}^{D''}(\delta_j'')_n}. \tag{A2}$$

We refer to the book of Srivastava and Manocha[8] for convergence and general properties of these series.

The $F^{(3)}$ function is defined as a generalization of Lauricella hypergeometric functions of three variables. Among some of the advantages of this notation, we note that all hypergeometric functions of three variables can be expressed in this way. For our purposes, we can restrict our attention to the function

$$F^{(3)}\left[\begin{matrix}-;\beta,\beta',-;\gamma,\gamma',\gamma''\\ -;\zeta,\zeta',-;-\end{matrix}\middle| x,y,z\right] = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\beta)_{l+n}(\beta')_{m+n}(\gamma)_l(\gamma')_m(\gamma'')_n}{(\zeta)_{l+n}(\zeta')_{m+n}}\frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}. \tag{A3}$$

When $b_1=1$ (The case $b_2=1$ is symmetrical), we should select $\gamma=0$, $\zeta=\gamma''$ in (A1), which reduces to

$$F^{(3)}\left[\begin{matrix}-;ia_1,ia_2,-;0,b_2-1,1\\ -;1,b_2,-;-\end{matrix}\middle| z_1,z_2,x_0\right]=F_1(ia_2,b_2-1,ia_1,b_2,z_2,x_0). \tag{A4}$$

If we also select $b_2=1$, i.e., $\gamma'=0$, $\zeta'=\gamma''$ we have

$$F^{(3)}\left[\begin{matrix}-;ia_1,ia_2,-;0,0,1\\ -;1,1,-;-\end{matrix}\middle| z_1,z_2,x_0\right]= {}_2F_1(ia_1,ia_2,1,x_0). \tag{A5}$$

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